

THE MOVABLE CONE VIA INTERSECTIONS

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ABSTRACT. We characterize the movable cone of divisors using intersections against curves on birational models.

1. INTRODUCTION

Cones of divisors play an essential role in describing the birational geometry of a smooth complex projective variety X . A key feature of these cones is their interplay with cones of curves via duality statements. The dual of the nef cone and the pseudo-effective cone of divisors were determined by [Kle66] and [BDPP04] respectively. We consider the third cone commonly used in birational geometry: the movable cone of divisors.

Definition 1.1. Let X be a smooth projective variety over \mathbb{C} . The movable cone $\overline{\text{Mov}}^1(X) \subset N^1(X)$ is the closure of the cone generated by classes of effective Cartier divisors L such that the base locus of $|L|$ has codimension at least 2. We say a divisor is movable if its numerical class lies in $\overline{\text{Mov}}^1(X)$.

A natural candidate for the dual of the movable cone is the closure of the cone generated by irreducible curves that deform to cover a codimension 1 subset of X . Such a curve is said to be movable in codimension 1, or a mov^1 -curve. However, as demonstrated by [Pay06, Example 1], these two cones are not dual in general.

Nevertheless, Debarre and Lazarsfeld have asked whether one can formulate a duality statement for movable divisors and mov^1 -curves. This has been accomplished for toric varieties in [Pay06] and for Mori Dream Spaces in [Cho10] by taking other birational models of X into account. Our main theorem proves an analogous statement for all smooth varieties. The statement involves the divisorial Zariski decomposition $L = P_\sigma(L) + N_\sigma(L)$ where $P_\sigma(L)$ is the movable part of L and $N_\sigma(L)$ is the “fixed” part of L (see Definition 2.2).

Theorem 1.2. *Let X be a smooth projective variety over \mathbb{C} and let L be a pseudo-effective divisor. L is not movable iff there is a mov^1 -curve C on X and a birational morphism $\phi : Y \rightarrow X$ from a smooth variety Y such that*

$$(P_\sigma(\phi^*L) + \phi_*^{-1}N_\sigma(L)) \cdot \tilde{C} < 0$$

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where \tilde{C} is the strict transform of a generic deformation of C and ϕ_*^{-1} denotes the strict transform of a divisor.

There does not seem to be an easy way to translate Theorem 1.2 into a statement involving only intersections on X . This is a symptom of the fact that the natural operation on movable divisors is the push-forward and not the pull-back.

Conceptually, the term $P_\sigma(\phi^*L) + \phi_*^{-1}N_\sigma(L)$ in Theorem 1.2 represents the “movable pullback” of L . For convenience we codify this as a definition.

Definition 1.3. Let X be a smooth projective variety over \mathbb{C} and let L be a pseudo-effective divisor on X . Suppose that $\phi : Y \rightarrow X$ is a birational map from a smooth variety Y . The movable transform of L on Y is defined to be

$$\phi_{\text{mov}}^{-1}(L) := P_\sigma(\phi^*L) + \phi_*^{-1}N_\sigma(L)$$

where ϕ_*^{-1} denotes the strict transform.

Note that the movable transform is not linear and is only defined for pseudo-effective divisors. Since

$$\phi_{\text{mov}}^{-1}(L) = \phi^*L - N_\sigma(Y/X, L)$$

where $N_\sigma(Y/X, L)$ denotes the relative divisorial Zariski decomposition of [Nak04], the movable transform precisely discounts the “extra” contributions to non-movability given by pulling-back.

Example 1.4. For surfaces Theorem 1.2 reduces to the usual duality of the nef and pseudo-effective cones. It is instructive to verify Theorem 1.2 directly in this case. Let L be a pseudo-effective divisor on a smooth surface and write the Zariski decomposition of L as $L = P + N$. There may be a component E of N such that $L \cdot E > 0$ (see [BDPP04] 6.3 Remark). However, choose a birational map $\phi : S' \rightarrow S$ such that the components of the strict transform $\phi_*^{-1}N$ are disjoint. As $\phi_*^{-1}N \leq N_\sigma(\phi^*L)$, any component \tilde{E} of $\phi_*^{-1}N$ satisfies

$$(P_\sigma(\phi^*L) + \phi_*^{-1}N) \cdot \tilde{E} < 0$$

by the orthogonality of the Zariski decomposition and the negative-definiteness of the self-intersection matrix of $N_\sigma(\phi^*L)$.

Example 1.5. If L is effective then $\phi_{\text{mov}}^{-1}(L)$ is not far from the strict transform of L . For example, the two divisors coincide when $P_\sigma(L)$ does not contain any ϕ -exceptional center.

When X is a smooth Mori dream space, the movable and strict transforms coincide for a divisor L general in its \mathbb{Q} -linear equivalence class. Furthermore, the movable transform of L on any sufficiently high birational model of X is the pull-back of a divisor on a small modification of X . Thus we recover the statements of [Pay06] and [Cho10]: for a smooth toric variety or Mori Dream Space X , a divisor class is movable iff its strict transform class on every small modification X' has non-negative intersection with every mov^1 -curve on X' .

In fact, the proof of Theorem 1.2 yields a slightly more precise statement. Recall that the restricted base locus $\mathbf{B}_-(L)$ measures the base locus of a divisor L perturbed by small ample divisors:

$$\mathbf{B}_-(L) = \bigcup_{A \text{ ample}} \left(\bigcap_{\substack{D \equiv L+A \\ D \geq 0}} \text{Supp}(D) \right).$$

Corollary 1.6. *Let X be a smooth projective variety over \mathbb{C} and let L be a pseudo-effective divisor. Suppose that V is an irreducible subvariety of X contained in $\mathbf{B}_-(L)$ and let $\psi : X' \rightarrow X$ be a smooth birational model resolving the ideal sheaf of V . Then there is a birational morphism $\phi : Y \rightarrow X'$ from a smooth variety Y and an irreducible curve \tilde{C} on Y such that*

$$\phi_{\text{mov}}^{-1}(\psi^*L) \cdot \tilde{C} < 0$$

and $\psi \circ \phi(\tilde{C})$ deforms to cover V .

In particular, although $\mathbf{B}_-(L)$ may not be covered by curves that have negative intersection with L , Corollary 1.6 shows that this is true in some birational sense.

Remark 1.7. Debarre and Lazarsfeld have also asked about the relationship between the closure of the cone of divisors whose stable base locus has codimension k and the closure of the cone of irreducible curves that deform to cover a codimension $k+1$ subset (for $0 < k < \dim X$).

Corollary 1.6 yields a birational formulation of this duality. For toric varieties and for $2 < k < \dim X$, [Pay06, Theorem 1] proves a slightly stronger statement. The main difference is that in [Pay06] it is unnecessary to first resolve V . It is unclear whether this stronger version should hold in general.

2. BACKGROUND

Throughout X will denote a smooth projective variety over \mathbb{C} . The term “divisor” will always refer to an \mathbb{R} -Cartier divisor unless otherwise qualified. The volume of a divisor L is

$$\text{vol}_X(L) = \limsup_{m \rightarrow \infty} \frac{h^0(X, \lfloor mL \rfloor)}{m^{\dim X}}.$$

2.1. Divisorial Zariski decomposition.

Definition 2.1 ([Nak04]). Let L be a pseudo-effective divisor on X . For any prime divisor E on X define

$$\sigma_E(L) = \max_{A \text{ ample}} \left(\min_{\substack{D \equiv L+A \\ D \geq 0}} \text{ord}_E(D) \right).$$

For any pseudo-effective L , [Nak04] shows that there are only finitely many prime divisors E on X with $\sigma_E(L) > 0$. Thus we can make the following definition.

Definition 2.2 ([Nak04]). Let L be a pseudo-effective divisor on X . Define

$$N_\sigma(L) = \sum \sigma_E(L)E \quad P_\sigma(L) = L - N_\sigma(L)$$

The decomposition $L = N_\sigma(L) + P_\sigma(L)$ is called the *divisorial Zariski decomposition* of L .

For our purposes, the key fact about the decomposition is that $P_\sigma(L)$ is a movable divisor ([Nak04]), so its restriction to any prime divisor E is pseudo-effective.

2.2. Numerical dimension and orthogonality. Given a pseudo-effective divisor L , the numerical dimension $\nu(L)$ of [Nak04] and [BDPP04] is a numerical measure of the “positivity” of L . There is also a restricted variant $\nu_{X|V}(L)$ introduced in [BFJ09]; since the definition is somewhat involved, we will only refer to a special subcase using an alternate characterization from [Leh10].

Definition 2.3. Let L be a pseudo-effective divisor on X . Fix a prime divisor E on X and choose $L' \equiv L$ whose support does not contain E . We say $\nu_{X|E}(L) = 0$ if

$$\liminf_{\phi} \text{vol}_{\tilde{E}}(P_\sigma(\phi^*L')|_{\tilde{E}}) = 0$$

where $\phi : \tilde{X} \rightarrow X$ varies over all birational maps and \tilde{E} denotes the strict transform of E .

The connection with geometry is given by the following (slightly weaker) version of the orthogonality theorem of [BDPP04] and [BFJ09].

Theorem 2.4. *Let L be a pseudo-effective divisor. If a prime divisor $E \subset X$ is contained in $\text{Supp}(N_\sigma(L))$ then $\nu_{X|E}(L) = 0$.*

3. PROOF

The proof of Theorem 1.2 is to reinterpret the orthogonality theorem of [BDPP04] using the techniques of [Leh10].

Proof of Theorem 1.2: Suppose that L is not movable. Denote by E a fixed divisorial component of $N_\sigma(L)$.

Fix a sufficiently general ample divisor A on X and choose ϵ small enough so that E is a component of $N_\sigma(L + \epsilon A)$. Applying the orthogonality theorem of [BDPP04], we see there is a birational map $\phi : Y \rightarrow X$ so that

- (1) \tilde{E} is smooth.
- (2) $\text{vol}_{\tilde{E}}(P_\sigma(\phi^*(L + \epsilon A))|_{\tilde{E}}) < \text{vol}_E(A|_E) = \text{vol}_{\tilde{E}}(\phi^*A|_{\tilde{E}})$.
- (3) The strict transform of every component of $N_\sigma(L)$ is disjoint.

There is a unique expression

$$P_\sigma(\phi^*(L + \epsilon A)) = P_\sigma(\phi^*L) + \phi^*A + \alpha(\epsilon)\tilde{E} + F$$

where \tilde{E} is the strict transform of E , F is an effective divisor with $F \leq N_\sigma(\phi^*L)$ and the support of F does not contain E , and $\alpha(\epsilon)$ is positive and goes to 0 as ϵ goes to 0. By shrinking ϵ we may ensure that $\alpha(\epsilon) < \sigma_E(L)$.

Condition (2) above, along with Lemma 3.1, show that the restriction $(P_\sigma(\phi^*L) + \alpha(\epsilon)\tilde{E})|_{\tilde{E}}$ is not pseudo-effective for any $\epsilon > 0$. Since $\alpha(\epsilon) < \sigma_E(L)$, we also have that $(P_\sigma(\phi^*L) + \sigma_E(L)\tilde{E})|_{\tilde{E}}$ is not pseudo-effective. As the strict transform of components of $N_\sigma(L)$ are disjoint, the restriction of $P_\sigma(\phi^*L) + \phi_*^{-1}N_\sigma(L)$ to \tilde{E} is still not pseudo-effective.

By [BDPP04, 0.2 Theorem] there is a movable curve \tilde{C} on \tilde{E} such that

$$(P_\sigma(\phi^*L) + \phi_*^{-1}N_\sigma(L)) \cdot \tilde{C} < 0.$$

Since \tilde{E} is not ϕ -exceptional, $C = \phi(\tilde{C})$ is a mov^1 -curve.

Conversely, if L is movable, then $\phi_{\text{mov}}^{-1}(L) = P_\sigma(\phi^*L)$ is also movable for every ϕ . Thus every movable transform has non-negative intersection with the strict transform of every mov^1 -curve general in its family. \square

Lemma 3.1. *Let X be a smooth projective variety and let L and L' be pseudo-effective divisors on X . Then $\text{vol}_X(L + L') \geq \text{vol}_X(L)$.*

Proof. We may assume L is big since otherwise the inequality is automatic. Then for any sufficiently small $\epsilon > 0$ we have

$$\text{vol}_X(L + L') = \text{vol}_X((1 - \epsilon)L + (\epsilon L + L')) \geq (1 - \epsilon)^{\dim X} \text{vol}_X(L)$$

since $\epsilon L + L'$ is big. \square

Proof of Corollary 1.6: Let E be the ψ -exceptional divisor dominating V . Since $E \subset \text{Supp}(N_\sigma(\psi^*L))$, we may argue as in the proof of Theorem 1.2 for ψ^*L and E to find a birational map ϕ such that $\phi_{\text{mov}}^{-1}(\psi^*L)|_{\tilde{E}}$ is not pseudo-effective.

[BDPP04, 2.4 Theorem] shows that there is some curve \tilde{C} on \tilde{E} with $\phi_{\text{mov}}^{-1}(\psi^*L) \cdot \tilde{C} < 0$ such that \tilde{C} deforms to cover \tilde{E} and is not contracted by any morphism from \tilde{E} . Choosing \tilde{C} on \tilde{E} to satisfy this stronger property, we obtain the statement of Corollary 1.6. \square

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